COMPUTING LINEAR ENVELOPES FOR UNIFORM B–SPLINE CURVES

DAVID LUTTERKORT

Full Paper

ABSTRACT. We derive a new, efficiently computable bound on the distance between a uniform spline and its B–Spline control polygon in terms of the second differences of the control points. The bound is piecewise linear and sharp for quadratic and cubic splines and decreases by a factor of 4 under uniform refinement. The bound is the basis for constructing envelopes for planar parametric curves.

1. INTRODUCTION

B-splines are the standard representation of smooth non-linear geometry in numerical calculations. The central feature that allows reasoning about the underlying nonlinear piecewise polynomials is the fact that the spline is closely outlined by the *control polygon*, a broken line connecting the control points. The efficiency of many applications hinges on an *a priori* estimate of the distance from the spline to its control polygon. For rendering, a refined control polygon is rendered instead of the B–spline itself. For intersecting two B–spline curves, an efficient and robust technique is to recursively refine and intersect their control polygons [HL89]. Assessing the exactness of these operations requires an *a priori*, uniform, linear bound on the distance of the B–spline and its (refined) control polygon. However, apart from the convex hull, no tight quantitative estimates exist to date.

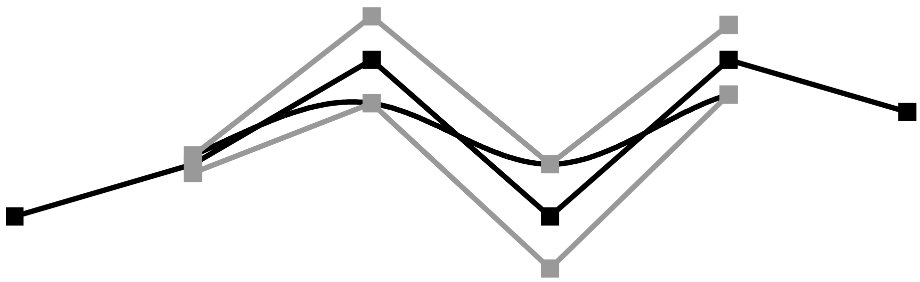

FIGURE 1. A cubic function *(black)* and its control points *(black squares)*. The envelope *(grey)* is defined by Theorem 3.

This paper introduces two new quantitative bounds for uniform B–splines that can be computed more efficiently than convex hulls. The first bounds the maximal distance between a spline of degree d and its B-spline control polygon in terms of second differences of the control points and at most $d-1$ constants. Each constant is obtained by evaluating one B–spline that only depends on the degree and not on the control points of the curve. The constants sum to

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$(d+1)/24$ which provides a second, even simpler, bound. The bound decreases by a factor of 4 when the spacing of the uniform knots is halved. Both bounds are piecewise linear with breaks at the corners of the control polygon and are *sharp for quadratic and cubic splines* in the sense that the distance from any corner of the control polygon to the spline is matched exactly by the bound.

This paper derives these bounds for *functions* and establishes the convergence of the bound under uniform refinement; the bounds are then applied to curves to obtain *localized* envelopes.

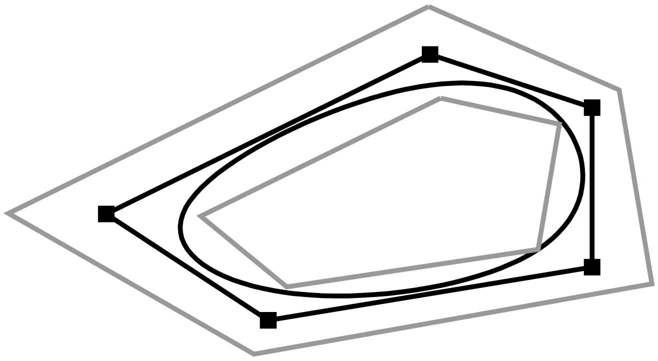

FIGURE 2. A cubic curve *(black)* and its control points *(black squares)*. The envelope *(grey)* is constructed with the bound from Theorem 3.

2. NOTATION

A piecewise polynomial p of degree d is a *uniform B–spline*, if it can be represented as

$$p = \sum_{k \in \mathbb{Z}} b^k N_k^d$$

where the real numbers b^k are the *control points* and the uniform B–spline basis functions N_k^d are defined over the uniform *knot sequence* (t_k), $t_k = k$, through the well–known recursion formulas for the B–Spline basis, see e.g. [dB93]. For simplicity, we assume that both the control point sequence and the knot sequence are biinfinite. The uniform B–spline basis functions are all shifts of one another and can be written as $N_j^d(\cdot) = N_0^d(\cdot - j) = N^d(\cdot - j)$. N^d is non-negative and has support only over the interval $[0, d+1)$.

The *Greville abscissae* t_k^* for the knot sequence $t_k = k$ are given by

$$t_k^* = \frac{1}{d} \sum_{i=k+1}^{k+d} t_i = k + \frac{d+1}{2}.$$

We denote the line segment from (t_k^*, a_1) to (t_{k+1}^*, a_2) by

$$\mathcal{L}_k(t \mid a_1, a_2) = a_1(t_{k+1}^* - t) + a_2(t - t_k^*)$$

and the linear interpolant of a function f over this interval by $\mathcal{L}_k(f) = \mathcal{L}_k(\cdot \mid f(t_k^*), f(t_{k+1}^*))$.

The *control polygon* ℓ of p is the piecewise linear interpolant of the control points b^k at the Greville abscissae t_k^*. Over the interval $[t_k^*, t_{k+1}^*]$, it is

$$\ell(t) = \mathcal{L}_k(t \mid b^k, b^{k+1}).$$

The (centered) second differences of b are defined as

$$\Delta_2 b^i = b^{i-1} - 2b^i + b^{i+1}.$$

Theorem 1 will show that only a few $\Delta_2 b^i$ influence the difference between p and ℓ at t_k^*. We denote the maximum in absolute value of all such second differences by $\|\Delta_2 b\|_k$,

$$\|\Delta_2 b\|_k = \max\left\{ \left|\Delta_2 b^i\right| \mid k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor \right\}.$$

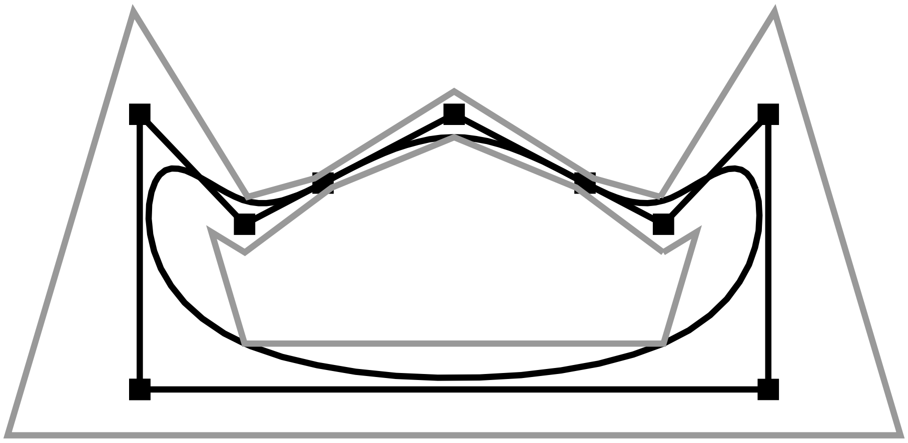

FIGURE 3. A cubic curve *(black)* and its control points *(black squares)*. The envelope *(grey)* is constructed with the bound from Theorem 3.

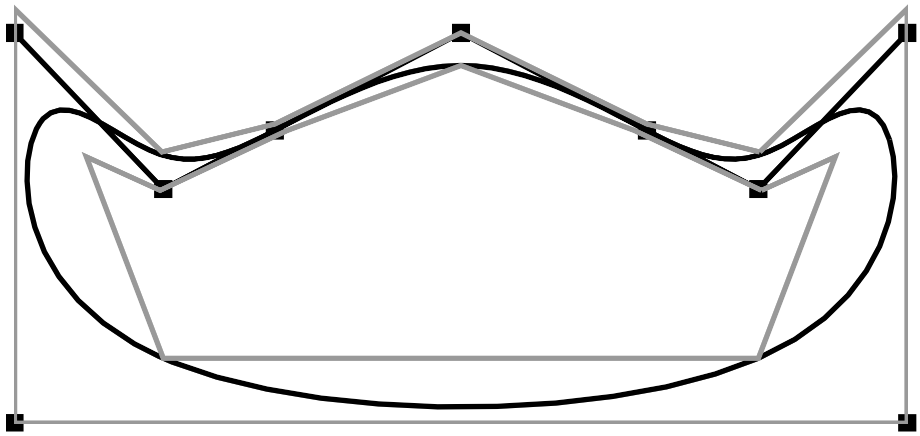

FIGURE 4. The curve *(black)* from Figure 3 enlarged with tighter envelope *(grey)* constructed from the bounds in Theorem 2.

A pivotal role in determining bounds is played by the functions β_{ki},

$$\beta_{ki} = \begin{cases} \sum_{j=-\infty}^{i} (i-j) N_j^d & i \leq k \\ \sum_{j=i}^{\infty} (j-i) N_j^d & i > k. \end{cases} \tag{2.1}$$

They capture the inherent contribution of the uniform B–spline basis to the bound between spline and control polygon.

3. Uniform B–splines Bounds

The key observation for deriving the bounds is that the difference between a uniform B–spline p and its control polygon can be factored into two parts: the second differences of the control polygon and the splines β_{ki}, which are independent of p. This observation is made precise in Theorem 1.

Theorem 1. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is given by*

$$p - \ell = \sum \Delta_2 b^i \, \beta_{ki}. \tag{3.2}$$

The functions β_{ki} are non-negative and convex on the interval $[t_k^*, t_{k+1}^*]$.
$\beta_{ki}(t_k^*) > 0$ if and only if $k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor$.

Proof. We write $p - \ell_k$ over $[t_k^*, t_{k+1}^*]$ as $\sum_i b^i \, \alpha_{ki}$, whith

$$\alpha_{ki}(t) = N_i^d(t) - \mathcal{L}_k(t \mid [i = k], [i = k+1])$$

where $[P] = 1$ if P is true and $[P] = 0$ otherwise.

We define the functions $\tilde{\beta}_{ki}$ as

$$\tilde{\beta}_{ki}(t) = \sum_{j=-\infty}^{i} (i-j) \, \alpha_{kj}.$$

The $\tilde{\beta}_{ki}$ are identical to the β_{ki} defined in equation (2.1): the partition of unity $\sum_j N_j^d = 1$ implies that $\sum_j \alpha_{kj} = 0$. The linear precision of B–Splines, $\sum_j t_j^* N_j^d = t$ implies

$$\sum_j j \, \alpha_{kj} = \sum_j t_j^* \, \alpha_{kj} - \sum_j \frac{d+1}{2} \, \alpha_{kj} = t - \mathcal{L}_k(t \mid t_k^*, t_{k+1}^*) - 0 = 0.$$

This means that $\sum_j (i-j) \, \alpha_{kj} = 0$ for any i and that $\tilde{\beta}_{ki} = \beta_{ki}$.

The β_{ki} are second anti–differences of the α_{ki} in the sense that $\alpha_{ki} = \Delta_2 \beta_{ki}$. We have

$$\beta_{ki} - \beta_{k,i-1} = \sum_{j=-\infty}^{i} (i-j) \, \alpha_{kj} - \sum_{j=-\infty}^{i-1} (i-1-j) \, \alpha_{kj} = \sum_{j=-\infty}^{i-1} \alpha_{kj}$$

and therefore

$$\Delta_2 \beta_{ki} = (\beta_{k,i+1} - \beta_{ki}) - (\beta_{k,i} - \beta_{k,i-1}) = \sum_{j=-\infty}^{i} \alpha_{kj} - \sum_{j=-\infty}^{i-1} \alpha_{kj} = \alpha_{ki}.$$

Summation by parts then proves the first part of the claim:

$$p - \ell_k = \sum b^i \, \alpha_{ki} = \sum b^i \, \Delta_2 \beta_{ki} = \sum \Delta_2 b^i \, \beta_{ki}.$$

Equation (2.1) shows that the B–spline coefficients of the β_{ki} are non-negative. The functions β_{ki} themselves are therefore non-negative.

The convexity of the β_{ki} follows from the convexity of their B–spline control polygons: for $i > k$, the control polygon of β_{ki} over $[t_k^*, t_{k+1}^*]$ is the control polygon of $t - i$, except that the negative control points have been replaced by zeros, resulting in a monotonically increasing and convex control polygon. For $i \leq k$, the control polygon of β_{ki} over $[t_k^*, t_{k+1}^*]$ is the control polygon of $i - t$ with negative control points replaced by zeros.

$\beta_{ki}(t_k^*)$ is positive only if one of the B–spline basis functions N_j^d in the summation (2.1) is positive at t_k^*. This is the case only for $\underline{k} = k - \lfloor d/2 \rfloor \leq j \leq k + \lfloor d/2 \rfloor = \overline{k}$, so that (2.1) can be written as

$$\beta_{ki}(t_k^*) = \begin{cases} \displaystyle\sum_{j=\underline{k}}^{i} (i-j)\, N_j^d(t_k^*) & i \leq k \\ \displaystyle\sum_{j=i}^{\overline{k}} (j-i)\, N_j^d(t_k^*) & i > k. \end{cases}$$

Furthermore, since $\underline{k} < k < \overline{k}$, we have that $\beta_{k\underline{k}}(t_k^*) = \beta_{k\overline{k}}(t_k^*) = 0$. This proves that $\beta_{ki}(t_k^*) > 0$ if and only if $k - \lfloor d/2 \rfloor \leq j \leq k + \lfloor d/2 \rfloor$. □

Theorem 1 immediately gives us a piecewise linear bound on $p - \ell$:

Theorem 2. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is bounded by*

$$\mathcal{L}_k\left(\sum \Delta_2^- b^i\, \beta_{ki}\right) \leq p - \ell \leq \mathcal{L}_k\left(\sum \Delta_2^+ b^i\, \beta_{ki}\right). \tag{3.3}$$

where $\Delta_2^+ b^i = \max\{\Delta_2 b^i, 0\}$ *and* $\Delta_2^- b^i = \min\{\Delta_2 b^i, 0\}$.

Proof. We have from Theorem 1

$$p - \ell = \sum_i \Delta_2 b^i\, \beta_{ki} = \sum_i \Delta_2^+ b^i\, \beta_{ki} + \sum_i \Delta_2^- b^i\, \beta_{ki}.$$

The positivity of the β_{ki} implies that the first sum on the right–hand side is positive and the second is negative and that therefore

$$\sum_i \Delta_2^- b^i\, \beta_{ki} \leq p - \ell \leq \sum_i \Delta_2^+ b^i\, \beta_{ki}.$$

Since the β_{ki} are convex over $[t_k^*, t_{k+1}^*]$, they can be bounded linearly to yield (3.3). □

An even simpler bound can be derived by subsuming the values of all the β_{ki} into one constant, $(d+1)/24$.

Theorem 3. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is bounded by*

$$|p - \ell| \leq \frac{d+1}{24}\, \mathcal{L}_k(\,\cdot \mid \|\Delta_2 b\|_k, \|\Delta_2 b\|_{k+1}). \tag{3.4}$$

Proof. By Theorem 1 and the convexity of the β_{ki} over $[t_k^*, t_{k+1}^*]$, we have

$$\begin{aligned} |p - \ell_k| &\leq \sum_i \left|\Delta_2 b^i\right| \beta_{ki} \leq \mathcal{L}_k\left(\sum_i \left|\Delta_2 b^i\right| \beta_{ki}\right) \\ &\leq \mathcal{L}_k(\,\cdot \mid \|\Delta_2 b\|_k \sum_i \beta_{ki}(t_k^*), \|\Delta_2 b\|_{k+1} \sum_i \beta_{ki}(t_{k+1}^*)). \end{aligned} \tag{3.5}$$

The theorem follows if we can show that $\sum_i \beta_{ki} \leq (d+1)/24$: over the interval $[t^*_k, t^*_{k+1}]$, only finitely many β_{ki} are non-negative. Let $\underline{k}$ and $\overline{k}$ be the smallest and largest index i such that $\beta_{ki} > 0$ on some part of $[t^*_k, t^*_{k+1}]$. We then have

$$\begin{aligned}\sum_i \beta_{ki} &= \sum_{i=\underline{k}}^{\overline{k}} \sum_{j=\underline{k}}^{i} (i-j)\,\alpha_{kj} = \sum_{j=\underline{k}}^{\overline{k}} \sum_{i=j}^{\overline{k}} (i-j)\,\alpha_{kj} = \sum_j \alpha_{kj} \sum_{i=0}^{\overline{k}-j} i \\ &= \sum_j \binom{j-\overline{k}}{2} \alpha_{kj} = \sum_j \binom{j}{2} \alpha_{kj} = \sum_j \binom{j-k}{2} N^d_j =: z.\end{aligned}$$

z is a quadratic polynomial, regardless of the degree of p. We express it in monomial form using the formula in the appendix:

$$\begin{aligned}z(t) &= t^2/2 - (k + \frac{d+2}{2})t + \frac{1}{8}(d+2k)^2 + \frac{1}{24}(24k + 13d + 10) \\ &= \frac{1}{2}\left(t^2 - (t^*_k + t^*_{k+1})t + \left(\frac{t^*_k + t^*_{k+1}}{2}\right)^2 + \frac{d-2}{12}\right).\end{aligned}$$

z attains its maximum over $[t^*_k, t^*_{k+1}]$ at one of the endpoints of the interval since it is a positive and convex function. Its values there are

$$z(t^*_k) = z(t^*_{k+1}) = \frac{d+1}{24}$$

and hence $z(t) = \sum_i \beta_{ki} \leq (d+1)/24$ for all $t \in [t^*_k, t^*_{k+1}]$. $\square$

4. Computing the bounds

The number of β_{ki} that are positive at t^*_k is $d-1$ for even degrees d and $d-2$ for odd d. This means in particular that for quadratic and cubic uniform B–splines only β_{kk} is positive at t^*_k and no B–spline evaluation is needed to compute the values of the $\beta_{ki}(t^*_k)$ for quadratic and cubic uniform B–splines: the proof of Theorem 3 showed that $\sum_i \beta_{ki} = (d+1)/24$ so that $\beta_{kk}(t^*_k) = (d+1)/24$ for quadratics and cubics.

To compute the bounds from Theorem 2 for degrees $d > 3$, we evaluate the functions β_{ki} at the Greville abscissa t^*_k for all $k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor$. It suffices to do this for $k = 0$. This requires $2\lfloor d/2 \rfloor - 1$ B–spline evaluations. The constants $\beta_{0i}(t^*_k)$ do only depend on d, not on the specific spline we wish to bound and can therefore be computed beforehand.

This means that both the bounds from Theorem 2 and Theorem 3 can be computed very efficiently; computing the bound from Theorem 2 at t^*_k requires one second difference calculation, $2\lfloor d/2 \rfloor - 1$ multiplications of $\Delta_2 b^i$ with $\beta_{ki}(t^*_k)$ and $2\lfloor d/2 \rfloor - 2$ additions. Computing the bound from Theorem 3 at t^*_k requires calculating one second difference, $2\lfloor d/2 \rfloor - 1$ absolute values and $2\lfloor d/2 \rfloor - 2$ comparisons to determine $\|\Delta_2 b\|_k$ and one multiplication with $(d+1)/24$.

The bound (3.4) from Theorem 3 is an equality at the t^*_k for quadratics and cubics, since only one $\beta_{ki}(t^*_k)$ is positive: the only inequalities in the proof of Theorem 3 are in equation (3.5), which are equalities at t^*_k if they contain only one summand.

5. Uniform refinement

An important operation on B–Splines is the refinement of the knot sequence or knot insertion. Knot insertion changes the representation of the piecewise polynomial p over the original knot sequence to one over a larger knot sequence and reduces the distance between spline and control polygon.

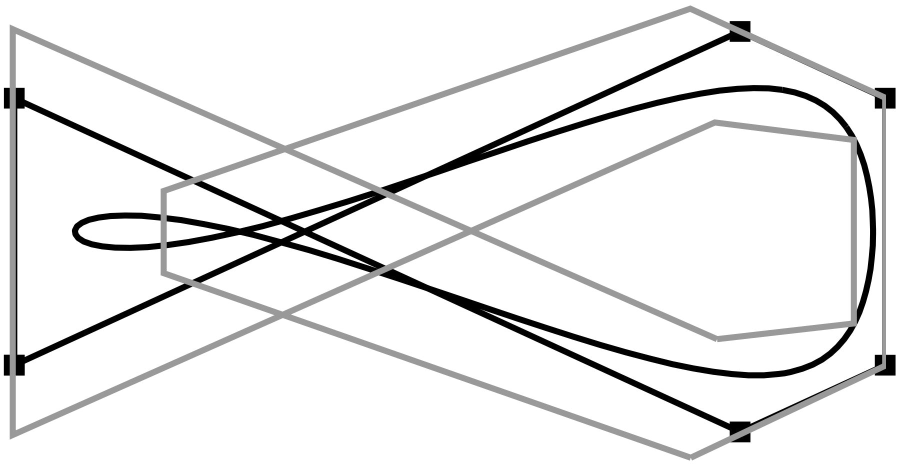

FIGURE 5. A self–intersecting quartic curve *(black)* and its control points *(black squares)*. The envelope *(grey)* is constructed with the bound from Theorem 2.

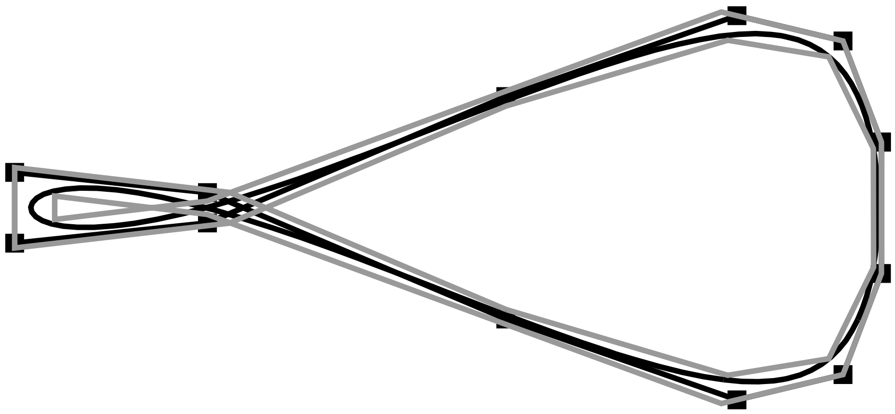

FIGURE 6. The quartic *(black)* and envelope *(grey)* from Figure 5 after one step of uniform refinement as described in Section 5. The envelope converges rapidly to the curve.

For uniform B–Splines, we consider the refinement of the knot sequence $t_k = k$ to the sequence $\hat{t}_k = k/2$, halving the distance between knots. We now have two different representations of p,

$$p(t) = \sum_k b^k N_k^d(t) = \sum_k \hat{b}_k N_k^d(2t).$$

The new control points $\hat{b}_k$ are given by

$$\hat{b}_{2i} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j} b^{i-j}, \qquad \hat{b}_{2i+1} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j+1} b^{i-j}. \tag{5.6}$$

Lemma 4. *The second differences $\Delta_2 \hat{b}_i$ of the new control polygon are related to the second differences $\Delta_2 b^i$ of the old control polygon through*

$$2^d \Delta_2 \hat{b}_{2i} = \sum_j \binom{d-1}{2j-1} \Delta_2 b^{i-j}$$
$$2^d \Delta_2 \hat{b}_{2i+1} = \sum_j \binom{d-1}{2j} \Delta_2 b^{i-j}.$$

Proof. The second derivative p'' of p is given by

$$p''(t) = \sum_k \Delta_2 b^{k-1} N_k^{d-2}(t) = \sum_k \Delta_2 \hat{b}_{k-1} N_k^{d-2}(2t)$$

which means that the $\Delta_2 \hat{b}_i$ can be obtained from the $\Delta_2 b_i$ through the formulas (5.6): we get for the even indices

$$\Delta_2 \hat{b}_{2i} = 2^{-d} \sum_{j=1}^{\lceil d/2 \rceil} \binom{d-1}{2j-1} \Delta_2 b^{i-j}$$

and for the odd indices

$$\Delta_2 \hat{b}_{2i+1} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil - 1} \binom{d-1}{2j} \Delta_2 b^{i-j}$$

□

Theorem 5.

$$\max_i \left| \Delta_2 \hat{b}_i \right| = \frac{1}{4} \max_i \left| \Delta_2 b^i \right|.$$

Proof. This is a direct consequence of the fact that $\sum_j \binom{d-1}{j} = 2^{d-1}$ and the symmetry of the binomial coefficients, which imply that

$$\sum_j \binom{d-1}{2j} = \sum_j \binom{d-1}{2j-1} = 2^{d-2}.$$

□

This relationship yields *a priori* estimates on the number of subdivisions needed to bring spline and control polygon within a given distance ε.

Theorem 5 does *not* imply that the bound from Theorem 3 decreases monotonically under uniform refinement. It is possible that the values $\|\Delta_2 b\|_k$ *increase* for some k. It does assure us though that the absolutely largest second difference decreases by a factor of 4. This ensures that the bound from Theorem 3 converges to 0 under repeated uniform refinement.

5.1. **Examples.** For quadratic B–Splines, uniform refinement is called Chaikin's algorithm and

$$\hat{b}_{2i} = 2^{-2}(3b^{i-1} + b^i) \qquad \hat{b}_{2i+1} = 2^{-2}(b^{i-1} + 3b^i).$$

This yields

$$\Delta_2 \hat{b}_{2i} = \Delta_2 \hat{b}_{2i-1} = \frac{1}{4} \Delta_2 b^{i-1}.$$

Since *every* second difference is guaranteed to decrease by a factor of four, the bound from Theorem 3 will indeed converge monotonically to 0 in this case.

Similarly, for cubic B–Splines we have

$$\hat{b}_{2i} = 2^{-3}(b^{i-2} + 6b^{i-1} + b^i) \qquad \hat{b}_{2i+1} = 2^{-3}(4b^{i-1} + 4b^i).$$

and

$$\Delta_2\hat{b}_{2i} = \frac{1}{4}\Delta_2 b^{i-1} \quad \text{and} \quad \Delta_2\hat{b}_{2i+1} = \frac{1}{8}(\Delta_2 b^{i-1} + \Delta_2 b^i).$$

Because of the second equality, one large second difference "bleeds" into its neighbors under refinement.

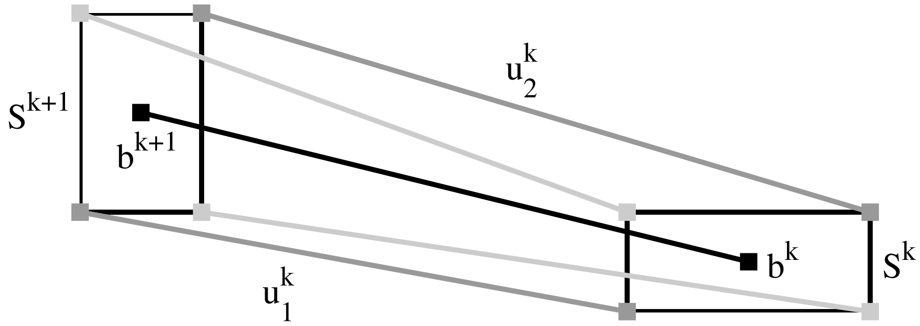

FIGURE 7. Constructing the envelope of a curve from the bounding rectangles S^k and S^{k+1}: only the two lines $\mathbf{u}_1^k$ and $\mathbf{u}_2^k$ are part of the convex hull of $S^k \cup S^{k+1}$ and are part of the envelope (see Section 6). The lines $\mathbf{v}_i^k$ are shown in dark and light grey.

6. Enveloping planar curves

Although it is possible to envelope uniform B–spline curves in *any* dimension in a manner similar to the one outlined below, we restrict ourselves to planar curves since this eases notation considerably without losing much of the essence of the more general case.

A planar parametric curve $\mathbf{p}$ is in uniform B–spline form if

$$\mathbf{p} = \sum_j \mathbf{b}^j N_j^d$$

where the $\mathbf{b}^j = (\mathbf{b}_0^j, \mathbf{b}_1^j) \in \mathbb{R}^2$ are the control points of $\mathbf{p}$ and the uniform B–spline basis N_j^d is defined as in Section 2. The curve $\mathbf{p}$ is closed if the control point sequence $(\mathbf{b}^j)$ is periodic.

The functional bounds are applied componentwise to parametric curves. It is convenient to write the bounds from Theorems 2 and 3 for functions p more abstractly as

$$\underline{\varrho}(t) \le p(t) - \ell(t) \le \overline{\varrho}(t) \qquad \text{for } t \in [t_k^*, t_{k+1}^*]. \tag{6.7}$$

For curves $\mathbf{p}$, the bound in the i–th component is denoted by $\underline{\varrho}_i \le \mathbf{p}_i - \ell_i \le \overline{\varrho}_i$.

By (6.7), $\mathbf{p}(t_k^*)$ is located in the axis–aligned rectangle S^k,

$$S^k = \left\{ \mathbf{x} \mid \underline{\varrho}_i(t_k^*) \le \mathbf{x}_i - \mathbf{b}_i^k \le \overline{\varrho}_i(t_k^*) \quad i = 0, 1 \right\}.$$

The point $\mathbf{p}(t)$, $t \in [t_k^*, t_{k+1}^*]$, lies in a rectangle $S(t)$, which by the linearity of $\underline{\varrho}$ and $\overline{\varrho}$ is a convex combination of S^k and S^{k+1},

$$S(t) = \mathcal{L}_k(t \mid S^k, S^{k+1}).$$

Over the interval $[t_k^*, t_{k+1}^*]$, the curve $\mathbf{p}$ is contained in the union of all $S(t)$, $t \in [t_k^*, t_{k+1}^*]$, which is the convex hull H^k of the corners of S^k and S^{k+1}.

Let $\mathbf{v}_i^k$, $i = 1, \ldots, 4$, be the line segments connecting corresponding corners of S^k and S^{k+1}; that means $\mathbf{v}_1^k$ connects the lower left corner of S^k to the lower left corner of S^{k+1}, $\mathbf{v}_2^k$ connects the lower right corner of S^k to the lower right corner of S^{k+1} etc. (see Figure 7)

H^k consists of parts of the boundaries of S^k and S^{k+1} and exactly two additional line segments $\mathbf{u}_1^k$ and $\mathbf{u}_2^k$ chosen from the $\mathbf{v}_i^k$. Since $\mathbf{u}_1^k$ and $\mathbf{u}_2^k$ are part of the convex hull H^k, they do not intersect the interiors of S^k and S^{k+1}. We do not need to actually compute intersections of the $\mathbf{v}_i^k$ and S^k, S^{k+1} to find $\mathbf{u}_1^k$ and $\mathbf{u}_2^k$ since S^k and S^{k+1} are axis–aligned; it suffices to compute the signs of the slopes of the $\mathbf{v}_i^k$. The $\mathbf{u}_i^k$ are separated by the line from $\mathbf{b}^k$ to $\mathbf{b}^{k+1}$; we call the one lying to the left of this line $\mathbf{u}_1^k$ and the one lying to the right of this line $\mathbf{u}_2^k$.

The sets $U_i = \{\mathbf{u}_i^k\}$ are not yet polylines: consecutive line segments $\mathbf{u}_i^k$ and $\mathbf{u}_i^{k+1}$ may intersect or not touch at all. We could turn the W_i into polylines by adding pieces of the boundaries of the S^k; this has the drawback that we might end up with four line segments in $W_1 \cup W_2$ for each control point of $\mathbf{p}$.

Instead, we modify the line segments in U_i to obtain a proper polyline W_i by extending (or shortening) the *line* $\mathbf{u}_i^k$ to start at its intersection with the line $\mathbf{u}_i^{k-1}$ and to end at its intersection with the line $\mathbf{u}_i^{k+1}$. Note that, as lines, any $\mathbf{u}_i^k$ intersects $\mathbf{u}_i^{k+1}$, guaranteeing that the above modifications will indeed yield a proper polyline W_i and that the sets H^k will be contained in the envelope bounded by W_1 to the left and by W_2 to the right. In this manner, we obtain polylines W_1 and W_2 that each contain exactly one line segment for each control point of $\mathbf{p}$.

The polylines W_1 and W_2 are a *local* envelope of $\mathbf{p}$: the curve–piece $\mathbf{p}([t_k^*, t_{k+1}^*])$ lies in the quadrangle spanned by the k–th pieces of W_1 and W_2.

Representing t^2 as a uniform B–Spline

In general, monomials can be represented as B–Splines through[1]

$$\binom{d}{j} t^j = \sum_k \frac{1}{d!} \left(\sum_{\sigma \in \Pi_d} \prod_{i=1}^{j} t_{k+\sigma(i)} \right) N_k^d$$

Here, Π_d is the set of all permutations of the numbers $\{1, \ldots, d\}$.

For uniform B–Splines and $j = 2$, this boils down to:

$$\begin{aligned}
\binom{d}{2} t^2 &= \frac{1}{2} \sum_k \sum_{\sigma \in \Pi_d} (t_{k+\sigma(1)} t_{k+\sigma(2)}) N_k^d \\
&= \sum_k \sum_{\sigma \in \Pi_d} ((k + \sigma(1))(k + \sigma(2))) N_k^d \\
&= \frac{1}{2} \sum_k \left(\sum_{p=1}^{d} \sum_{q=1}^{d} (k+p)(k+q) - \sum_{p=1}^{d} (k+p)^2 \right) N_k^d \\
&= \frac{1}{2} \sum_k \left[\frac{d^2}{4} (2k + d + 1)^2 - \frac{d}{6} (6k(k + d + 1) + 2d^2 + 3d + 1) \right] N_k^d \\
&= \sum_k \frac{1}{12} \binom{d}{2} (3(d + 2k)^2 + 5d + 2 + 12k) N_k^d.
\end{aligned}$$

[1] This is a corrected version of a formula given in [Gol95]. The original formula is missing the factor $1/d!$ on the right hand side.

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David Lutterkort, Dept. of Computer Science, Purdue University, West Lafayette, IN 47907–1398
E-mail address: `lutter@cs.purdue.edu`