COMPUTING LINEAR ENVELOPES FOR UNIFORM B–SPLINE CURVES

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Extended Abstract

ABSTRACT. We derive a new, efficiently computable bound on the distance between a uniform spline and its B–Spline control polygon in terms of the second differences of the control points. The bound is piecewise linear and sharp for quadratic and cubic splines and decreases by a factor of 4 under uniform refinement. The bound is the basis for constructing envelopes for planar parametric curves.

1. MOTIVATION & NOTATION

B-splines are the standard representation of smooth non-linear geometry in numerical calculations. The central feature that allows reasoning about the underlying nonlinear piecewise polynomials is the fact that the spline is closely outlined by the *control polygon*, a broken line connecting the control points. The efficiency of many applications hinges on an *a priori* estimate of the distance from the spline to its control polygon. For rendering, a refined control polygon is rendered instead of the B–spline itself. For intersecting two B–spline curves, an efficient and robust technique is to recursively refine and intersect their control polygons [HL89]. Assessing the exactness of these operations requires an *a priori*, uniform, linear bound on the distance of the B–spline and its (refined) control polygon. However, apart from the convex hull, no tight quantitative estimates exist to date.

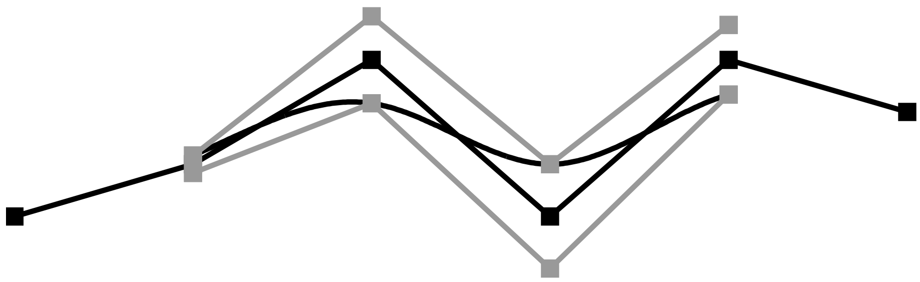

FIGURE 1. A cubic function *(black)* and its control points *(black squares)*. The envelope *(grey)* is defined by Theorem 3.

This paper introduces two new quantitative bounds for uniform B–splines that can be computed more efficiently than convex hulls. The first bounds the maximal distance between a spline of degree d and its B-spline control polygon in terms of second differences of the control points and at most $d-1$ constants. Each constant is obtained by evaluating one B–spline that only depends on the degree and not on the control points of the curve. The constants sum to

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$(d+1)/24$ which provides a second, even simpler, bound. The bound decreases by a factor of 4 when the spacing of the uniform knots is halved. Both bounds are piecewise linear with breaks at the corners of the control polygon and are *sharp for quadratic and cubic splines* in the sense that the distance from any corner of the control polygon to the spline is matched exactly by the bound.

The paper derives these bounds for *functions* and establishes the convergence of the bound under uniform refinement; the bounds are then applied to curves to obtain *localized* envelopes.

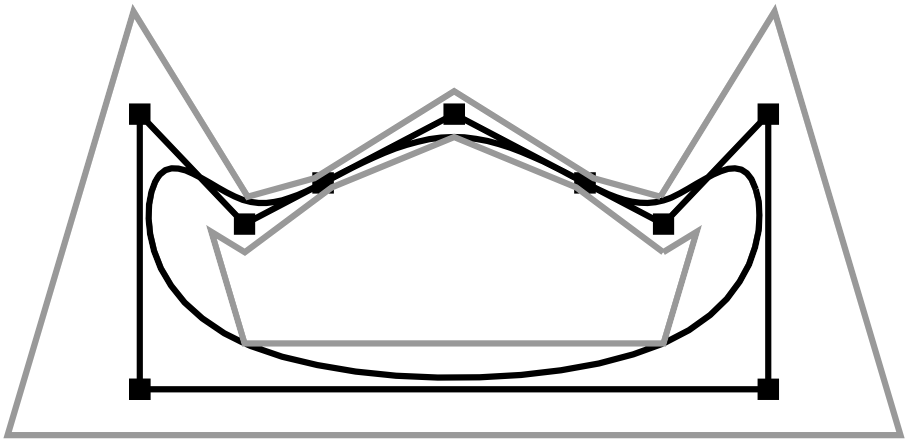


FIGURE 2. A cubic curve *(black)* and its control points *(black squares)*. The envelope *(grey)* is constructed with the bound from Theorem 3.

1.1. **Notation.** A piecewise polynomial p of degree d is a *uniform B–spline*, if it can be represented as

$$p = \sum_{k \in \mathbb{Z}} b^k N_k^d$$

where the real numbers b^k are the *control points* and the uniform B–spline basis functions N_k^d are defined over the uniform *knot sequence* (t_k), $t_k = k$, through the well–known recursion formulas for the B–Spline basis, see e.g. [dB93]. For simplicity, we assume that both the control point sequence and the knot sequence are biinfinite.

The *Greville abscissae* t_k^* for the knot sequence $t_k = k$ are given by $t_k^* = \mathrm{k} + (\mathrm{d}+1)/2$.

We denote the line segment from (t_k^*, a_1) to (t_{k+1}^*, a_2) by

$$\mathcal{L}_k(t \mid a_1, a_2) = a_1(t_{k+1}^* - t) + a_2(t - t_k^*)$$

and the linear interpolant of a function f over this interval by $\mathcal{L}_k(f) = \mathcal{L}_k(\cdot \mid f(t_k^*), f(t_{k+1}^*))$.

The *control polygon* ℓ of p is the piecewise linear interpolant of the control points b^k at the Greville abscissae t_k^*. Over the interval $[t_k^*, t_{k+1}^*]$, it is

$$\ell(t) = \mathcal{L}_k(t \mid b^k, b^{k+1}).$$

The (centered) second differences of b are defined as

$$\Delta_2 b^i = b^{i-1} - 2b^i + b^{i+1}.$$

Only a few $\Delta_2 b^i$ influence the difference between p and ℓ at t_k^*. We denote the maximum in absolute value of all such second differences by $\|\Delta_2 b\|_k$,

$$\|\Delta_2 b\|_k = \max\left\{ \left|\Delta_2 b^i\right| \mid k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor \right\}.$$

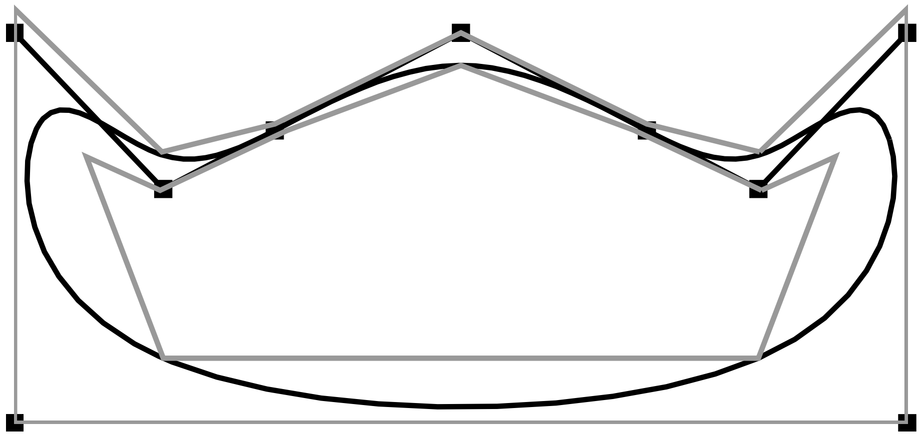


FIGURE 3. The curve *(black)* from Figure 2 enlarged with tighter envelope *(grey)* constructed from the bounds in Theorem 2.

A pivotal role in determining bounds is played by the functions β_{ki},

$$\beta_{ki} = \begin{cases} \displaystyle\sum_{j=-\infty}^{i} (i-j) N_j^d & i \le k \\ \displaystyle\sum_{j=i}^{\infty} (j-i) N_j^d & i > k. \end{cases} \tag{1.1}$$

They capture the inherent contribution of the uniform B–spline basis to the bound between spline and control polygon.

2. Uniform B–splines Bounds

The key observation for deriving the bounds is that the difference between a uniform B–spline p and its control polygon can be factored into two parts: the second differences of the control polygon and the splines β_{ki}, which are independent of p. This observation is made precise in Theorem 1.

Theorem 1. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is given by*

$$p - \ell = \sum \Delta_2 b^i \, \beta_{ki}. \tag{2.2}$$

The functions β_{ki} are non-negative and convex on the interval $[t_k^*, t_{k+1}^*]$.
$\beta_{ki}(t_k^*) > 0$ if and only if $k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor$.

Theorem 1 immediately gives us a piecewise linear bound on $p - \ell$:

Theorem 2. *Over the interval* $[t_k^*, t_{k+1}^*]$, *the difference between a uniform B–spline* p *and its control polygon* ℓ *is bounded by*

$$\mathcal{L}_k\left(\sum \Delta_2^- b^i\, \beta_{ki}\right) \leq p - \ell \leq \mathcal{L}_k\left(\sum \Delta_2^+ b^i\, \beta_{ki}\right). \tag{2.3}$$

where $\Delta_2^+ b^i = \max\{\Delta_2 b^i, 0\}$ *and* $\Delta_2^- b^i = \min\{\Delta_2 b^i, 0\}$.

An even simpler bound can be derived by subsuming the values of all the β_{ki} into one constant, $(d+1)/24$.

Theorem 3. *Over the interval* $[t_k^*, t_{k+1}^*]$, *the difference between a uniform B–spline* p *and its control polygon* ℓ *is bounded by*

$$|p - \ell| \leq \frac{d+1}{24}\, \mathcal{L}_k(\,\cdot \mid \|\Delta_2 b\|_k, \|\Delta_2 b\|_{k+1}). \tag{2.4}$$

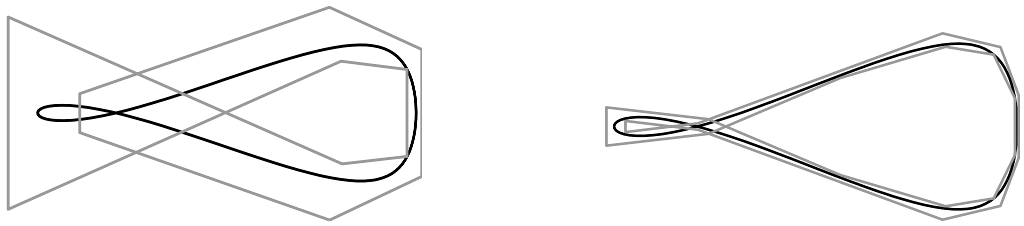
3. Computing the bounds

The number of β_{ki} that are positive at t_k^* is $d-1$ for even degrees d and $d-2$ for odd d. This means in particular that for quadratic and cubic uniform B–splines only β_{kk} is positive at t_k^* and no B–spline evaluation is needed to compute the values of the $\beta_{ki}(t_k^*)$ for quadratic and cubic uniform B–splines: they are $\beta_{kk}(t_k^*) = 3/24$ for quadratic and $\beta_{kk}(t_k^*) = 4/24$ for cubic uniform B–splines.

To compute the bounds from Theorem 2 for degrees $d > 3$, we evaluate the functions β_{ki} at the Greville abscissa t_k^* for all $k - \lfloor d/2 \rfloor < i < k + \lfloor d/2 \rfloor$. It suffices to do this for $k = 0$. This requires $2\lfloor d/2 \rfloor - 1$ B–spline evaluations. The constants $\beta_{0i}(t_k^*)$ do only depend on d, not on the specific spline we wish to bound and can therefore be computed beforehand.

This means that both the bounds from Theorem 2 and Theorem 3 can be computed very efficiently; computing the bound from Theorem 2 at t_k^* requires one second difference calculation, $2\lfloor d/2 \rfloor - 1$ multiplications of $\Delta_2 b^i$ with $\beta_{ki}(t_k^*)$ and $2\lfloor d/2 \rfloor - 2$ additions. Computing the bound from Theorem 3 at t_k^* requires calculating one second difference, $2\lfloor d/2 \rfloor - 1$ absolute values and $2\lfloor d/2 \rfloor - 2$ comparisons to determine $\|\Delta_2 b\|_k$ and one multiplication with $(d+1)/24$.

The bound (2.4) from Theorem 3 is an equality at the t_k^* for quadratics and cubics, since only one $\beta_{ki}(t_k^*)$ is positive.



Input curve After one step of uniform refinement

Figure 4. A self–intersecting quartic curve *(black)* and its envelope *(grey)*, constructed with the bound from Theorem 2.

4. Uniform refinement

An important operation on B–Splines is the refinement of the knot sequence or knot insertion. Knot insertion changes the representation of the piecewise polynomial p over the original knot sequence to one over a larger knot sequence and reduces the distance between spline and control polygon.

For uniform B–Splines, we consider the refinement of the knot sequence $t_k = k$ to the sequence $\hat{t}_k = k/2$, halving the distance between knots. We now have two different representations of p,

$$p(t) = \sum_k b^k N_k^d(t) = \sum_k \hat{b}_k N_k^d(2t).$$

The second differences of the control points $\hat{b}^j$ are related to those of the control points b^j by $\max_i \left|\Delta_2 \hat{b}_i\right| = \max_i \left|\Delta_2 b^i\right|/4$. This relationship yields *a priori* estimates on the number of subdivisions needed to bring spline and control polygon within a given distance ε.

5. Enveloping planar curves

Although it is possible to envelope uniform B–spline curves in *any* dimension in a manner similar to the one outlined below, we restrict ourselves to planar curves since this eases notation considerably without losing much of the essence of the more general case.

A planar parametric curve $\mathbf{p}$ is in uniform B–spline form if $\mathbf{p} = \sum_j \mathbf{b}^j N_j^d$ where the $\mathbf{b}^j = (\mathbf{b}_0^j, \mathbf{b}_1^j) \in \mathbb{R}^2$ are the control points of $\mathbf{p}$ and the uniform B–spline basis N_j^d is defined as in Section 1.1. The curve $\mathbf{p}$ is closed if the control point sequence $(\mathbf{b}^j)$ is periodic.

The functional bounds are applied componentwise to parametric curves. It is convenient to write the bounds from Theorems 2 and 3 for functions p more abstractly as

$$\underline{\varrho}(t) \le p(t) - \ell(t) \le \overline{\varrho}(t) \qquad \text{for } t \in [t_k^*, t_{k+1}^*]. \tag{5.5}$$

For curves $\mathbf{p}$, the bound in the i–th component is denoted by $\underline{\varrho}_i \le \mathbf{p}_i - \ell_i \le \overline{\varrho}_i$.

By (5.5), $\mathbf{p}(t_k^*)$ is located in the axis–aligned rectangle S^k,

$$S^k = \left\{ \mathbf{x} \mid \underline{\varrho}_i(t_k^*) \le \mathbf{x}_i - \mathbf{b}_i^k \le \overline{\varrho}_i(t_k^*) \quad i = 0, 1 \right\}.$$

The point $\mathbf{p}(t)$, $t \in [t_k^*, t_{k+1}^*]$, lies in a rectangle $S(t)$, which by the linearity of $\underline{\varrho}$ and $\overline{\varrho}$ is a convex combination of S^k and S^{k+1},

$$S(t) = \mathcal{L}_k(t \mid S^k, S^{k+1}).$$

Over the interval $[t_k^*, t_{k+1}^*]$, the curve $\mathbf{p}$ is contained in the union of all $S(t)$, $t \in [t_k^*, t_{k+1}^*]$, which can be efficiently computed to obtain a *local* envelope for $\mathbf{p}([t_k^*, t_{k+1}^*])$. The details of this construction can be found in the accompanying paper.

References

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